

Electromagnetics from a quasistatic perspective

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Quasistatics is introduced so that it fits smoothly into the standard textbook presentation of electrodynamics. The usual path from statics to general electrodynamics is rather short and surprisingly simple. A closer look reveals however that it is not without confusing issues as has been illustrated by many contributions to this Journal. Quasistatic theory is conceptually useful by providing an intermediate level in between statics and the full set of Maxwell's equations. Quasistatics is easier than general electrodynamics and in some ways more similar to statics. It is however, in terms of interesting physics and important applications, far richer than statics. Quasistatics is much used in electromagnetic modeling, an activity that today is possible on a PC and which also has great pedagogical potential. The use of electromagnetic simulations in teaching gives additional support for the importance of quasistatics. This activity may also motivate some change of focus in the presentation of basic electrodynamics.

I. INTRODUCTION

Applications of electrodynamics may be in the *static*, *quasistatic* or general *high frequency* regime. Quasistatics is more or less neglected in most textbooks and the purpose of this paper is to present material that can fill this gap in a course on the level of for example Griffith's textbook.¹ There are good reasons to do so including:

1. The step from statics to general electrodynamics is huge in terms of its physical content. Quasistatic models are useful by providing intermediate levels in the theory. Thereby several confusing issues that result from the very condensed standard derivation of Maxwell's equations may be avoided. Questions concerning Coulomb's and Biot-Savart's laws in non-static situations should be addressed.^{2,3,4} So should also the appearance of dynamic electric fields in regions where the magnetic field seems to be absent (like outside a long solenoid or toroidal coil with a time-varying current).^{5,6,7,8,9} Another confusing issue follows from the standard textbook motivation for the displacement current where it is introduced in order to make the equations of electrodynamics consistent with charge conservation. The physical impact of this term is remarkable and includes in particular electromagnetic waves in free space. But why should waves in *free space without charges and currents* follow from *charge conservation*?¹⁰ Quasistatics is useful in the discussion of these and many other questions.
2. There are plenty of interesting phenomena within the quasistatic regime. Using the full set of Maxwell's equations are many times unnecessarily complicated since these equations can describe the most intricate electromagnetic wave phenomena involving short time-scales or high frequency. Such an analysis may be difficult and is not necessary in quasistatic situations.¹¹
3. Many real world applications involve the numeri-

cal solution of the time-dependent Maxwell equations in three dimensions. However, quite often one is interested in phenomena where some quasistatic analysis is sufficient. This amounts to the replacement of an hyperbolic model with an elliptic (or parabolic) one which can be solved in more economic ways.^{12,13,14}

4. Basic quasistatic theory may be introduced in an elementary and simple way suitable for basic courses in electrodynamics. Like static theory also quasistatics may be given two equivalent formulations. The first is in terms of force laws, including straightforward generalizations of the static Coulomb and Biot-Savart laws. The second formulation is in terms of simple approximations in Maxwell's differential equations.

This paper is organized as follows. Basic theory of quasistatics is presented in Section II. Subsections A and B contain the two equivalent formulations of quasistatics, one in terms of force laws and the other in terms of approximations in Maxwell's differential equations. A salient feature of quasistatics is instantaneous interaction at a distance. In subsection C the corresponding $c \rightarrow \infty$ limit of general electrodynamics is used to derive quasistatics. Subsections D, E and F include quasistatic theory for the electromagnetic potentials, the fields from a moving point charge and the quasistatic Poynting theorems. Alternatives to the standard textbook derivation of Maxwell's equations are discussed in Section III from a quasistatic perspective.

Sections IV and V concern some confusing issues that have been discussed in many contributions to this Journal. In Section IV we consider the laws of Coulomb and Biot-Savart within general time-dependent theory.² Sufficient and necessary conditions for the exact validity of these laws are given. Section V contains comments on the old question #6 of this Journal which concerns the fields outside a solenoid with a time-varying current.⁵

Quasistatics is useful not only for providing improved understanding of basic theory but it is also important

for applications, in particular for numerical simulations. In Section VI we discuss the possibility of using electromagnetic simulations in basic courses and exemplify with quasistatic equations for eddy currents. A couple of PDE-solvers which have been used for electromagnetic modeling on a PC is mentioned. A summary is given in Section VII.

II. QUASISTATICS

Quasistatics within electrodynamics refers to a regime where "the system is small compared with the electromagnetic wavelength associated with the dominant time-scale of the problem".¹⁵ The fields are then propagated instantaneously so we are dealing with a kind of $c \rightarrow \infty$ limit. We consider in this paper three major *quasistatic models*. These are EQS (electroquasistatics)¹¹, MQS (magnetoquasistatics)¹¹ and the Darwin model^{12,13,14}. EQS includes capacitive but not inductive effects, MQS includes inductive but not capacitive effects while the Darwin model includes both capacitive and inductive effects. The Biot-Savart law is valid in all three models while the Coulomb law is valid only in EQS. In MQS and Darwin there is also, besides the Coulomb field, an additional contribution to the electric field due to magnetic induction. The source of this electric field is thus $\partial \mathbf{B}/\partial t$ in Faraday's law.¹⁶

A. From the laws of Coulomb, Biot-Savart and Faraday to the EQS, MQS and Darwin models.

Dynamical systems that proceed from one state to another as though they are static (at each fixed time) are commonly said to be *quasistatic*. For electromagnetism the static theory builds on the Coulomb and Biot-Savart laws together with the static continuity equation. Quasistatics would then simply be obtained by the allowance for time-dependence in the first two otherwise unchanged laws

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{r}', t) \hat{\mathbf{R}}}{R^2} d\tau' \quad (1)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \iiint \frac{\mathbf{J}(\mathbf{r}', t) \times \hat{\mathbf{R}}}{R^2} d\tau' \quad (2)$$

where we use the notations¹⁷

$$\mathbf{R} = \mathbf{r} - \mathbf{r}', \quad R = |\mathbf{R}|, \quad \hat{\mathbf{R}} = \mathbf{R}/R \quad (3)$$

It would however appear strange to keep the static continuity equation unchanged in this time-dependent situation so we replace it with the usual continuity equation¹⁸

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \quad (4)$$

The three equations (1), (2) and (4) constitute the EQS model that includes capacitive but not inductive effects. Charge may be accumulated in this model but this requires work and energy may as usual be associated with the electric field. Magnetic energy is however outside the scope of EQS because there is no magnetically induced electric field and accordingly no back emf. No work is then required to create a magnetic field by starting an electric current.

We now like to include electromagnetic induction. Then also $\partial \mathbf{B}/\partial t$ acts as a source of electric fields and the total electric field becomes the sum of two parts

$$\mathbf{E} = \mathbf{E}_C + \mathbf{E}_F \quad (5)$$

The first term is the Coulomb electric field

$$\mathbf{E}_C(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{r}', t) \hat{\mathbf{R}}}{R^2} d\tau' \quad (6)$$

and the second term is the Faraday electric field which may be defined by a Biot-Savart like integral expression as (cf. equation 2)¹⁹

$$\mathbf{E}_F(\mathbf{r}, t) = -\frac{1}{4\pi} \iiint \frac{\partial \mathbf{B}(\mathbf{r}', t)}{\partial t} \times \frac{\hat{\mathbf{R}}}{R^2} d\tau' \quad (7)$$

We note that \mathbf{E}_F solves the equations

$$\nabla \times \mathbf{E}_F = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{E}_F = 0 \quad (8)$$

while the corresponding equations for \mathbf{E}_C is

$$\nabla \times \mathbf{E}_C = 0, \quad \nabla \cdot \mathbf{E}_C = \frac{\rho}{\epsilon_0} \quad (9)$$

The expressions (6) and (7) are the unique solutions of (8) and (9) provided appropriate boundary conditions at infinity are used.²⁰

The Darwin model may be defined by the equations (2) and (4)-(7). This is a quasistatic model that includes both capacitive and inductive phenomena. Note that for given current and charge densities we directly obtain the electromagnetic fields in terms of integrals without the appearance of any time-retardation. Instead the integral expressions in the laws of Coulomb and Biot-Savart play an important role in this dynamical model.

The MQS model is obtained from Darwin if the usual continuity equation is replaced by the static one

$$\nabla \cdot \mathbf{J} = 0 \quad (10)$$

Thus the MQS model may be defined by the equations (2), (5), (6), (7) and (10). It is different from both EQS and Darwin by including inductive but not capacitive effects. A confusing feature of MQS is that the very fundamental continuity equation may be violated by MQS-solutions. Only stationary currents are allowed in MQS

and these cannot explain changes in the charge density, thus one cannot interpret the currents in MQS in terms of charge transportation. Of course, only sufficiently good approximations of solutions to the Maxwell equations are interesting in the real world so the continuity equation must still be almost true in some sense. Ampère's law, which imply equation (10), is valid in MQS but not in EQS or Darwin (see the next subsection). Thus Ampère's law is not always valid in quasistatics. Griffiths and Heald remark that "The application of Ampère's law in quasistatic situations can be an extremely delicate matter".²

The textbook of Haus and Melcher¹¹ builds up an understanding of electrodynamics by using both EQS and MQS. This is of particular significance in the relation between electromagnetic field theory and circuit theory. Then EQS involves capacitance features and MQS inductance features. For systems involving capacitance and inductance both models are needed. For such applications it is crucial that capacitive and inductive aspects are not both important in the same spatial place.²¹ The use of two complementary quasistatic models in the same physical system is clearly a complicating feature if we like to model the whole system numerically. We would then have to divide the whole spatial region into EQS and MQS subregions with appropriate continuity conditions at the interfaces. A better alternative may be to use the Darwin model which embrace all the physics contained in EQS and MQS, still being quasistatic.

B. From Maxwell's equations to EQS, MQS and Darwin.

Let us now consider the formulation of quasistatics in terms of differential equations. The starting point is general electrodynamics with the Maxwell equations:

$$\nabla \cdot \mathbf{E} = (1/\varepsilon_0) \rho \quad (11)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (12)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (13)$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \quad (14)$$

These equations are complete and general as they stand. However, in the presence of polarizable/magnetizable media it is in practice very convenient to *write* them in a different way by introducing the D- and the H-fields. Then only the free charge and current densities appear explicitly in the equations. For formal simplicity just E-

TABLE I: This table give two equivalent definitions of each of the quasistatic models EQS, MQS and Darwin

Model	Def.A, eq.#	Def.B, eq.#
EQS	(1),(2),(4)	(11),(12),(14),(15)
MQS	(2),(5),(6),(7),(10)	(11),(12),(13),(16)
Darwin	(2),(4),(5),(6),(7)	(11),(12),(13),(17), (9)

and B-fields will be used in the present paper but it is straightforward to modify the expressions so that a formulation with D- and H-fields is obtained.

Let us now formulate the quasistatic models EQS, MQS and Darwin as approximations of Maxwell's equations (11)-(14). The relations to the integral formulas in subsection A above will also be considered. The EQS model is obtained from the Maxwell equations simply by neglecting $\partial \mathbf{B}/\partial t$ in Faraday's law while MQS is obtained by instead neglecting $\partial \mathbf{E}/\partial t$ in the Ampère-Maxwell law. Thus in EQS we have instead of (13) the equation

$$\nabla \times \mathbf{E} = 0 \quad (15)$$

and in MQS the equation (14) is replaced by the usual Ampère law²²

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad (16)$$

To get the Darwin model we do not neglect all of $\partial \mathbf{E}/\partial t$ but keep the Coulomb part of the E-field (defined by equation (6) above) and replaces the Ampère-Maxwell law with the Ampère-Darwin equation²³

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}_C}{\partial t} \right) \quad (17)$$

Note an important difference between these two last equations. The Ampère law (16) implies equation (10) and may violate charge conservation in dynamic situations. The Ampère-Darwin equation, like the Ampère-Maxwell equation, is consistent with the continuity equation (4).

The electrostatic models may thus be defined with focus on the laws of Coulomb and Biot-Savart (like in the previous subsection) or alternatively as approximations of Maxwell's equations. These definitions are summarized in Table I. The proof that the A and B columns define the same models involves only standard procedures found in basic textbooks. In electrostatics one start from the Coulomb law and obtain the divergence and curl of the electric field. By the Helmholtz theorem we have the equivalence (assuming appropriate conditions at infinity²⁰)

$$(1) \iff (11) \text{ and } (15) \quad (18)$$

TABLE II: The table includes several familiar laws from electromagnetics as well as the Maxwell-Darwin equation and state if they are valid within the EQS, MQS and Darwin models.

Eq.#	Equation	EQS	MQS	Darwin
(1)	Coulomb's law	yes	no	no
(2)	Biot-Savart's law	yes	yes	yes
(4)	Continuity eq.	yes	no	yes
(10)	$\nabla \cdot \mathbf{J} = 0$	no	yes	no
(11)	Gauss law	yes	yes	yes
(12)	$\nabla \cdot \mathbf{B} = 0$	yes	yes	yes
(13)	Faraday's law	no	yes	yes
(14)	Ampère-Maxwell	yes	no	no
(15)	$\nabla \times \mathbf{E} = 0$	yes	no	no
(16)	Ampère's law	no	yes	no
(17)	Ampère-Darwin	yes	no	yes

In the same way we of course have (6) \Leftrightarrow (9). In magneto-statics one starts from the law of Biot-Savart and obtain the divergence and curl of the magnetic field. One finds the equivalence

$$(2) \text{ and } (10) \Leftrightarrow (12) \text{ and } (16) \quad (19)$$

The above relations trivially remain valid if we allow for time-dependence where time appears only as a parameter. However, in a time-dependent situation it is logical to use the general continuity equation (4) instead of the static one (10). Then instead of (19) we find

$$(2) \text{ and } (4) \Leftrightarrow (12) \text{ and } (17) \quad (20)$$

Thus the Ampère-Darwin equation is a mathematical implication of Biot-Savart's law combined with the continuity equation. The relation (20) is not included in most standard textbooks but has (more or less explicitly) appeared in several contributions to this Journal.^{24,25,26}

The equation (7) for the Faraday electric field is formally analogous to the Biot-Savart law (2) for the magnetic field. Formally similar to (19) is the equivalence

$$(7) \text{ and } (12) \Leftrightarrow (8) \text{ and } (12) \quad (21)$$

Table II includes some laws of electromagnetics and states if they are valid within EQS, MQS or Darwin. All the included equations are familiar from standard electromagnetics with the exception of Ampère-Darwin's equation (17).

C. A limit of instantaneous propagation in Maxwell's equations

In subsection A above we obtained the Darwin model from EQS by including Faraday's law in a simple and

straightforward way. An alternative procedure is to follow the standard textbook derivation of Maxwell's equations¹ but in the final step, when the displacement current is added to Ampère's law for consistency with the continuity equation, find the Ampère-Darwin equation (17) rather than the Ampère-Maxwell equation (14). This replaces the usual *maximal* assumption in the correcting term¹⁰ with a kind of *minimal* assumption. Both these derivations of Darwin's model may however seem rather superficial and arbitrary. It would be nice to obtain it from Maxwell's equations by using some more systematic method. A basic feature of quasistatic approximations to Maxwell's equations is the instantaneous propagation of fields. Thus it should be possible to consider quasistatics as some limit $c \rightarrow \infty$ of Maxwell's equations. This is however quite a singular limit and the procedure must be further specified. We take the absence of time-retardation as being the most salient property of quasistatics. Let us express Maxwell's equations in terms of the potentials (V, \mathbf{A}) so that the retarded time appears explicitly. We choose to use the Coulomb gauge

$$\nabla \cdot \mathbf{A} = 0 \quad (22)$$

and follow the derivation in Nielson and Lewis¹² (the reason for *not* using the Lorentz gauge is considered soon). The equations for the potentials become (see Griffiths¹ p. 421),

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} \quad (23)$$

$$\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \nabla \frac{\partial V}{\partial t} \quad (24)$$

Time-retardation appears explicitly if we solve equation (24) for the vector potential in terms of an integral in the usual way. The omission of retarded time in this integral is the same as excluding the second order time-derivative so that

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \nabla \frac{\partial V}{\partial t} \quad (25)$$

The model so obtained consists of the equations (22), (23) and (25). These constitute the Darwin model in terms of potentials.²⁷ The usual Darwin equations (11), (12), (13) and (17) is obtained by use of

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E}_C = -\nabla V \quad (26)$$

Why didn't we use the Lorentz gauge? It is straightforward to make the corresponding calculations also in that case. We then get a model approximating Maxwell's equations which seems pretty close to the Darwin model with the Ampère-Darwin law as one of the equations. However, Gauss law (11) is not obtained but a new term

appears in the corresponding equation. Certainly we prefer not to change Gauss law and this problem, as we have seen, does not appear if we use the Coulomb gauge in the approximation procedure. Using the Coulomb gauge in the Darwin model also have the nice consequence that the Coulomb part of the E-field is given by the scalar potential $\mathbf{E}_C = -\nabla V$ and the Faraday part by the vector potential $\mathbf{E}_F = -\partial\mathbf{A}/\partial t$.

D. The potential representations of EQS, MQS and Darwin

We use the Coulomb gauge. The magnetic field is written $\mathbf{B} = \nabla \times \mathbf{A}$ for all models. For MQS the vector potential then satisfy

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad (27)$$

or in integral form

$$\mathbf{A} = \frac{\mu_0}{4\pi} \iiint \frac{\mathbf{J}}{R} d\tau' \quad (28)$$

For EQS and Darwin the corresponding equations are (25) or

$$\mathbf{A} = \frac{\mu_0}{4\pi} \iiint \frac{(\mathbf{J} + \mathbf{J} \cdot \hat{\mathbf{R}} \hat{\mathbf{R}})}{2R} d\tau' \quad (29)$$

It takes some manipulation involving partial integration and the continuity equation to derive (29) from (25). The scalar potential is just the Coulomb potential for EQS, MQS and Darwin,

$$V = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho}{R} d\tau' \quad (30)$$

The electric field for EQS is conservative and is just the Coulomb field

$$\mathbf{E} = -\nabla V \quad (31)$$

while the electric field for MQS and Darwin also include the magnetically induced electric field so that

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad (32)$$

E. Fields of a moving point charge

We will now consider the quasistatic fields from a point charge in general motion. Coulomb's law is often taken as a starting point for electrostatics. It is conceptually simple to build on interactions between point charges in this way, so why not develop all of electrodynamics by generalizing this approach? This is discussed by Griffiths (chapter 2).¹ Two major problems are

1. The force between two point charges depends not only on their separation but also on both their velocities and accelerations
2. Furthermore, it is the position, velocity and acceleration at *retarded time* of the other particle that matters.

The second point is far more challenging than the first one. However, time-retardation vanishes within quasistatics and the approach with interacting point charges become quite simple and instructive. In the point charge approach to electrodynamic models we automatically include the continuity equation (4). Therefore this section concerns EQS and Darwin but not MQS. Let us start with the EQS model. The fields from a point charge Q within EQS is

$$\mathbf{E}(\mathbf{r}, t) = \frac{Q}{4\pi\epsilon_0} \frac{\hat{\mathbf{R}}}{R^2} \quad (33)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0 Q}{4\pi} \frac{\mathbf{v} \times \hat{\mathbf{R}}}{R^2} \quad (34)$$

where the notations in (3) are used. The position of Q is \mathbf{r}' which is a function of time and $\mathbf{v} = d\mathbf{r}'/dt$. From these equations we obtain (1) and (2) while the continuity equation (4) is implied by the point charge description.

Let us now consider the Darwin model. Then equation (34) remains valid while (33) only gives the Coulomb part of the electric field

$$\mathbf{E}_C(\mathbf{r}, t) = \frac{Q}{4\pi\epsilon_0} \frac{\hat{\mathbf{R}}}{R^2} \quad (35)$$

The magnetically induced part \mathbf{E}_F must now also be included. However, it is then convenient to start all over again using the electromagnetic potentials. The potentials are

$$V(\mathbf{r}, t) = \frac{Q}{4\pi\epsilon_0 R} \quad (36)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 Q}{4\pi} \frac{\mathbf{v} + \mathbf{v} \cdot \hat{\mathbf{R}} \hat{\mathbf{R}}}{2R} \quad (37)$$

The scalar potential (36) is just the Coulomb potential and the vector potential (37) may be found from (29). An alternative derivation is by the ansatz

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 Q}{4\pi} \frac{\mathbf{v}}{R} + \nabla \phi \quad (38)$$

where the scalar field ϕ is determined by the Coulomb gauge condition (22). It takes a little algebra to find

$$\phi = -\frac{\mu_0 Q}{8\pi} \mathbf{v} \cdot \hat{\mathbf{R}}$$

and thus (37).²⁸ The electromagnetic fields is related to the potentials in the usual way (26). The magnetically induced part of the electric field is

$$\mathbf{E}_F = -\partial\mathbf{A}/\partial t \quad (39)$$

This expression will obviously involve the acceleration $\mathbf{a} = d\mathbf{v}/dt$ of Q . The Darwin model should thus follow from the potentials (36) and (37) of a point charge and it is sufficient to check (2)-(7). Only equation (7) is not obvious and the easiest approach is to use the equivalent differential equations (8). But these equations are trivially satisfied by (39) using the Coulomb gauge and $\mathbf{B} = \nabla \times \mathbf{A}$.

The standard textbook approach to electrostatics begins with Coulomb's law (33) for a charge at rest. Sometimes the corresponding approach to magnetostatics is used by taking (34) as a starting point but now, of course, with a moving charge. This may in principle seem wrong since this is a time-dependent system. Most textbooks take this seriously and starts instead from the Biot-Savart law with stationary current. However, leaving the point particle point of view also makes magnetostatics more difficult than electrostatics. It comes as a relief that the magnetostatic differential equations (12) and (16) are formally similar to, and not much more difficult than, the electrostatic equations (11) and (15). Quasistatics is easier than statics in this respect; it allows for the use of (34).

A conventional Lagrangian description of the electromagnetic interaction between two or more charged particles is possible only if time-retardation may be neglected. It is straightforward to give such a formulation constructed from the potentials (36) and (37).³⁰ This results in the so called Darwin Lagrangian that was first obtained by Oliver Heaviside in 1891.^{31,32} It was found again by C.G. Darwin in 1920 by an expansion of the Lienard-Wiechert potentials.³³ Rather surprisingly it then turns out that that terms up to the order $(v/c)^2$ inclusive are to be kept in the Darwin Lagrangian (see Jackson¹⁵ p.596).

F. The Poynting theorem

Let us follow the standard derivation of Poynting's theorem while using the quasistatic models. I.e. we add the two equations obtained by taking the scalar product of the curl \mathbf{E} equation with the \mathbf{B} -field and the scalar product of the curl \mathbf{B} equation with the \mathbf{E} -field. We then get (after the usual manipulations) for EQS

$$\frac{\partial}{\partial t}(\frac{1}{2}\varepsilon_0 E^2) + \nabla \cdot (\frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}) = -\mathbf{E} \cdot \mathbf{J} \quad (40)$$

for MQS we obtain

$$\frac{\partial}{\partial t}(\frac{1}{2\mu_0} B^2) + \nabla \cdot (\frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}) = -\mathbf{E} \cdot \mathbf{J} \quad (41)$$

and for the Darwin model

$$\frac{\partial}{\partial t}(\frac{1}{2}\varepsilon_0 E_C^2 + \frac{1}{2\mu_0} B^2) + \nabla \cdot (\frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} + \varepsilon_0 \frac{\partial V}{\partial t} \frac{\partial \mathbf{A}}{\partial t}) = -\mathbf{E} \cdot \mathbf{J} \quad (42)$$

In EQS only the electric field is associated with energy. Building up a magnetic field costs no energy in this model due to the absence of a counteracting induced \mathbf{E} -field. In MQS only the magnetic field has energy. Changes of the electric field is associated with changes in the charge density. However, the continuity equation is not satisfied and there is no charge transport for which we could calculate the required energy. Finally the Darwin model includes both magnetic and electric energy. Note that only the Coulomb-part of the \mathbf{E} -field is associated with energy. An important qualitative difference between EQS and MQS on the one side and Darwin's model on the other is the possibility of natural resonances in the latter. Since Darwin includes both capacitive and inductive features we may in principle use these equations to model, for example, some field theoretic manifestation of a LC circuit.

III. IF MAXWELL HAD WORKED IN BETWEEN AMPÈRE AND FARADAY

We consider in this section various procedures to find the full set of Maxwell equations starting from the laws of Coulomb and Biot-Savart. The quasistatic perspective is useful and the models EQS, MQS and Darwin will appear as intermediate stages. The title of this section refers to the possibility of introducing the displacement current before Faraday's law.³⁴

Let us start with the standard textbook procedure of finding Maxwell's equations.¹ From the static laws of Coulomb and Biot-Savart we find, by use of the static continuity equation, the static limit of Maxwell's equations. By introducing magnetic induction we then obtain MQS (i.e. "Electrodynamics before Maxwell" in Griffiths textbook). The final step, motivated by the need for consistency with charge conservation (4), is to introduce the displacement current. At this point a question mentioned already in the introduction may surface: Maxwell's equations includes waves in *free space* but why should these follow from *charge conservation*? Quasistatics is useful for discussing this issue. It is indeed not necessary to introduce all of the displacement current to save charge conservation. It is sufficient to include the Coulomb part of it¹⁰ and then we get the Darwin model. At this point no surprising new physics appears. The step from Darwin to the Maxwell's equations may then, to begin with, be motivated by symmetry and beauty of equations. The explicit appearance of the Coulomb electric field in the Darwin model is a rather unsatisfactory feature and it is so easily fixed by just replacing it with the total electric field. Amazing new physics now appears and this is accompanied with beautiful new mathematical structure like the Lorentz invariance of both the Maxwell equations and of the trajectories of test charges. This is of course

surprising but, at least, the previous rather mystifying motivation for it has now been removed.

Jammer and Stachel discuss what might have happened if Maxwell had worked in between Ampère and Faraday.³⁴ As in the standard textbook derivation of Maxwell's equations one may first derive the static limit of Maxwell's equations from the laws of Coulomb and Biot-Savart combined with the static continuity equation. At this stage Maxwell might have added the displacement current before the discovery of Faraday's law. This would result in the EQS model which, according to Jammer and Stachel, is exactly Galilei invariant. They suggest in the article abstract that the discovery of Faraday "would have confronted physicists with the dilemma: give up the Galilean relativity principle for electromagnetism (ether hypothesis), or modify it (special relativity). This suggest a new pedagogical approach to electromagnetic theory, in which the displacement current and the Galilean relativity principle are introduced before the induction term is discussed." This approach is however less striking than it first seems. We like to include the appropriate invariance structure not only for the field equations but also for the trajectories of test charges. Here one get problems with the Galilei invariance of EQS while the Lorentz invariance of Maxwell's equations is perfect. The problem of finding satisfactory quasistatic and Galilei invariant approximations to Maxwell's equations is not an easy one.³⁵

A third approach to Maxwell equations is of more quasistatic nature. We avoid in this derivation the static limit of Maxwell's equation by allowing for (trivial) time-dependence in the laws of Coulomb (1) and Biot-Savart (2). We obtain the differential equations of EQS (i.e. the equations (11), (12), (14)) and (15) by pure mathematics if we also use the (time-dependent) continuity equation (4). Maxwell's equations now follow directly when the Faraday law is introduced. Also in this case (like in the standard textbook derivation) a question may appear. Why do the magnetic induction experiments of Faraday imply electromagnetic waves and time-retardation? The answer is somewhat more hidden in this case. The electric field in EQS is a pure Coulomb field \mathbf{E}_C but when we include magnetic induction we also get the Faraday electric field \mathbf{E}_F . It is now not clear if we should have \mathbf{E} or \mathbf{E}_C in (14). In the latter case (the *minimal* assumption) we get the Darwin model and no qualitatively new physics appears (only what is needed to explain the experiments of Faraday). The step from Ampère-Darwin to the Ampère-Maxwell law has already been discussed.

IV. THE LAWS OF COULOMB AND BIOT-SAVART IN TIME-DEPENDENT THEORY

The laws of Coulomb and Biot-Savart provide a starting point for the static theory of electromagnetics. Let us consider the following two questions:

1. How should the static laws of Coulomb and Biot

-Savart be generalized to time-dependent theory?

2. Consider the trivial generalization given by the equations (1) and (2) where time is included as a parameter. What is the significance of these formulas?

These questions are usually not addressed in the textbooks. The first question is considered by Jefimenko and the resulting formulas involves (as must be expected) integrals where the retarded time appears.²⁹ Jefimenko's generalized laws of Coulomb and Biot-Savart are used by Griffiths and Heald to address the second question. They find that²

- (a) The generalized Biot-Savart law reduces to the standard one if $\partial^2 \mathbf{J} / \partial t^2 = 0$ (then, from the continuity equation, also $\partial^3 \rho / \partial t^3 = 0$).
- (b) The generalized Coulomb law reduces to the standard one if $\partial \mathbf{J} / \partial t = 0$ (and thus also $\partial^2 \rho / \partial t^2 = 0$).
- (c) The law of Ampère holds if $\partial^2 \mathbf{J} / \partial t^2 = 0$ and $\partial \rho / \partial t = 0$.

These results are all sufficient conditions for the laws but only (c) is also necessary. Below necessary and sufficient conditions are formulated. It is natural to consider the validity of Coulomb and Biot-Savart in time-dependent theory from the perspective of quasistatics. We may divide electromagnetics into three regimes: statics, quasistatics and high frequency. However, statics is just a particular case of the general Maxwell equations while quasistatics is an approximation. This motivate a third question :

3. When does it happen that a solution in quasistatics is an exact solution to Maxwell's equations?

It will be shown that a solution of the Darwin model also solves Maxwell's equations if and only if the current has the form

$$\mathbf{J}(\mathbf{r}, t) = \mathbf{a}(\mathbf{r})t + \mathbf{b}(\mathbf{r}) - \varepsilon_0 \frac{\partial \mathbf{E}_C(\mathbf{r}, t)}{\partial t} \quad (43)$$

where the Coulomb field is defined by (6) and the vectorfields \mathbf{a} and \mathbf{b} satisfy $\nabla \cdot \mathbf{a} = \nabla \cdot \mathbf{b} = 0$. Formally we write this

$$\text{Maxwell} + \text{Biot-Savart} \iff \text{Darwin} + \text{Eq. (43)} \quad (44)$$

Let us compare this with (a) above. The last term in (43) may appear new and unfamiliar in the context of being a part of the true current (this term but with opposite sign is familiar as a part of the displacement current). However, this term is just the solenoidal part of the current. Thus consider any current density \mathbf{J} and write it uniquely in accordance with Helmholtz theorem as the sum of two parts $\mathbf{J} = \mathbf{J}_T + \mathbf{J}_L$ where \mathbf{J}_T is irrotational and \mathbf{J}_L is solenoidal.^{20,36} Then it follows from the

continuity equation and the time-derivative of (9) that $\mathbf{J}_L = -\varepsilon_0 \partial \mathbf{E}_C / \partial t$. The result (a) may be reformulated in a way similar to (44) but then we only get the left implication (i.e. " \Leftarrow ") and the last term in (43) is then assumed to be linear in time. The leaking capacitor is one of a few examples below showing that quasistatics may apply exactly also with a nonlinear time-dependence of the current.

Let us now prove (44) and start with the right implication. That Darwin is satisfied follows directly from the equivalence (20). Then both Ampère-Maxwell and Ampère-Darwin are satisfied implying $\partial \mathbf{E}_F / \partial t = 0$. From (8) it then follows that $\partial^2 \mathbf{B} / \partial t^2 = 0$. From the second order time-derivative of Ampère-Darwin and (12) we now get

$$\frac{\partial^2}{\partial t^2} \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}_C}{\partial t} \right) = 0 \quad (45)$$

and (43) easily follows (including the conditions $\nabla \cdot \mathbf{a} = \nabla \cdot \mathbf{b} = 0$). Consider next the left implication in (44). By substituting (43) in the Maxwell-Darwin equation we find $\nabla \times \mathbf{B} = \mu_0(\mathbf{a}t + \mathbf{b})$ so that \mathbf{B} is at most linear in t . From the time-derivative of (8) we then get $\partial \mathbf{E}_F / \partial t = 0$. In this case the Ampère-Darwin and the Ampère-Maxwell laws are the same and the solution to Darwin also solves Maxwell's equations.

Above we found a necessary and sufficient condition (44) for the exact validity of Biot-Savart's law within Maxwell's equations. Thereby the result (a) of Griffiths and Heald was generalized. Let us now consider the corresponding result for Coulomb's law and allow for a current of the form

$$\mathbf{J}(\mathbf{r}, t) = \mathbf{b}(\mathbf{r}) - \varepsilon_0 \frac{\partial \mathbf{E}_C(\mathbf{r}, t)}{\partial t} \quad (46)$$

where $\nabla \cdot \mathbf{b} = 0$. We will prove that

$$\text{Maxwell} + \text{Coulomb} \iff \text{EQS} + \text{Eq.(46)} \quad (47)$$

Let us start with the right implication in (47). From Coulomb's law it follows the E-field is conservative so that EQS is obtained from Maxwell. From Faraday's law it then follows that $\partial \mathbf{B} / \partial t = 0$ and from the time-derivative of Ampère-Maxwell we now get (using $\mathbf{E} = \mathbf{E}_C$)

$$\frac{\partial}{\partial t} \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}_C}{\partial t} \right) = 0 \quad (48)$$

and (46) easily follows. To obtain the left implication of (47) we first substitute (46) in Ampère-Maxwell and obtain. $\nabla \times \mathbf{B} = \mu_0 \mathbf{b}$. This equation together with (12) results in a time-independent B-field and thus Faraday's law is satisfied. This is what is needed for the Maxwell equations to be valid for an EQS solution. The proof of (47) is now completed and the result (b) is generalized.

Let us finally consider the result (c) involving Ampère's law. This condition turns out to be not only sufficient

but also necessary. Consider a current density of the form

$$\mathbf{J}(\mathbf{r}, t) = \mathbf{a}(\mathbf{r})t + \mathbf{b}(\mathbf{r}) \quad (49)$$

where the vectorfields \mathbf{a} and \mathbf{b} satisfy $\nabla \cdot \mathbf{a} = \nabla \cdot \mathbf{b} = 0$. Then

$$\text{Maxwell} + \text{Ampère} \iff \text{MQS} + \text{Eq.(49)} + \text{Eq.(4)} \quad (50)$$

Consider first the right implication. Both Ampère's law and Ampère-Maxwell are valid so $\partial \mathbf{E} / \partial t = 0$. By using the time-derivative of Faraday's law we then find $\partial^2 \mathbf{B} / \partial t^2 = 0$ so from the second order time-derived Ampère's law we get $\partial^2 \mathbf{J} / \partial t^2 = 0$. Then (49) follows if we also use that the current has zero divergence. The left implication in (47) is obtained by just reversing the procedure above.

We will now consider a few examples involving the results above. The first one will also be used in the next subsection.

Example 1 Consider an (infinitely) long solenoid or a toroidal coil with time-varying current $I(t)$. We assume the coils are wound so that we may neglect the axial current in the solenoid and the poloidal current in the toroid.

Within quasistatic we get (exactly) zero B-field outside the coils (from the Biot-Savart law and symmetry). The MQS and Darwin models coincide because there is no charge density and accordingly $\mathbf{E}_C = 0$. There is a magnetically induced electric field $\mathbf{E} = \mathbf{E}_F$ outside the coils in spite the fact that the B-field vanishes exactly. In general this is only an approximative solution to the Maxwell equations since the Ampère law (and of course the Ampère-Darwin law) rather than the Ampère-Maxwell law is satisfied outside the coils. An exact solution is obtained, as we have seen, when the current is linear in time. The electric field is then constant in time.

The second example shows that quasistatics may agree exactly with Maxwell's equations even when the time-variation is quite arbitrary.

Example 2 Consider any given spherically symmetric charge and current density satisfying the continuity equation (4).

By the spherical symmetry we get $\nabla \times \mathbf{E} = \nabla \times \mathbf{B} = 0$. The solution to EQS, Darwin and Maxwell's equations is $\mathbf{E} = \mathbf{E}_C$, $\mathbf{B} = 0$ and $\mathbf{J} = -\varepsilon_0 \partial \mathbf{E}_C / \partial t$ which is valid for any prescribed charge density $\rho = \rho(r, t)$ in (6). The equivalences (44) and (47) apply with $\mathbf{a} = \mathbf{b} = 0$. A particular case is a leaking spherical capacitor. We assume that the medium in between the spherical plates has some conductance and does not violate the spherical symmetry. In between the plates there is, during the slow discharge, zero charge density. The Ampère law is clearly not satisfied because $\mathbf{B} = 0$ while $\mathbf{J} \neq 0$.

The examples 3-5 below illustrates that in a linear, isotropic and homogeneous conductor the conduction current (assuming Ohm's law), in certain cases, does not

create any magnetic field. This corresponds to exact solutions of Maxwell's equations that also solves EQS and Darwin.

Example 3 *Let us now consider an example without symmetry in the charge distribution. Space is assumed to be linear, isotropic, conducting and homogeneous characterized by $(\epsilon_0, \mu_0, \sigma)$. Assume that we at time $t = 0$ know the charge density $\rho(\mathbf{r}, 0) = \rho_0(\mathbf{r})$.*

From $\mathbf{J} = \sigma \mathbf{E}$ in the continuity equation and by use of Gauss law we find

$$\rho(\mathbf{r}, t) = e^{-(\sigma/\epsilon_0)t} \rho_0(\mathbf{r}) \quad (51)$$

The corresponding solution to Maxwell's equation is

$$\mathbf{E}(\mathbf{r}, t) = e^{-(\sigma/\epsilon_0)t} \mathbf{E}_0(\mathbf{r}), \quad \mathbf{B} = 0 \quad (52)$$

where $\mathbf{E}_0(\mathbf{r})$ is the Coulomb field due to the charge density $\rho_0(\mathbf{r})$.

Example 4 *The last example above may be modified so that we are dealing with a finite conductor surrounded by free space. Let the conductor contain all given charge $\rho_0(\mathbf{r})$ at time $t = 0$ and let the surface of the conductor be an equipotential surface of $V_0(\mathbf{r})$ defined by*

$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho_0(\mathbf{r}')}{R} d\tau' \quad (53)$$

Inside the conductor both the charge distribution and the solution to Maxwell's equations is the same as in the previous example. Outside the fields are static with $\mathbf{E} = \mathbf{E}_0(\mathbf{r}) = -\nabla V_0$ and $\mathbf{B} = 0$. The surface charge density on the conductor is

$$\sigma_S(\mathbf{r}, t) = \epsilon_0 \left(1 - e^{-(\sigma/\epsilon_0)t} \right) \hat{\mathbf{n}}(\mathbf{r}) \cdot \mathbf{E}_0(\mathbf{r}) \quad (54)$$

where $\hat{\mathbf{n}}$ is the outward normal.

Example 5 *Related to the last example is the leaking capacitor of arbitrary geometry and with homogeneous, isotropic and weakly conducting material in between the two perfect conductors.⁴*

In this case we must define $V_0(\mathbf{r})$ by Laplace equation with appropriate boundary conditions on the two perfect conductors. The solution may be then expressed in terms of $\mathbf{E} = \mathbf{E}_0(\mathbf{r}) = -\nabla V_0$ as in the two last examples above.

V. A QUASISTATIC PERSPECTIVE ON QUESTION #6

A time-varying current of a long solenoid causes an induced electric field at the outside. But how can that be? There is no magnetic field at the outside! A.P French asked a similar question and it was addressed in several papers.^{5,6,7,8,9} The answers included two essential points

- The vanishing of the B-field is a static phenomena while in the nonstatic case there is a small magnetic field outside the solenoid.
- It is not good to consider the time-varying electromagnetic field as a source for the induced electric field, the source is rather the time-varying current.¹⁶

However, even though both these statements are correct in view of the Maxwell equations, there may remain some uncomfortable feelings. The way we in practice calculate the induced electric field from Faraday's law make it natural to think in terms of the time-varying B-field as a source and, even if there is a small B-field outside the solenoid, *the main part of this source is well separated in space from the effect we consider* (i.e. the induced outside E-field). This apparent action at a distance seems to contradict the local action quality of Maxwell's equations. But of course, a careful analysis shows that there is no true contradiction involved.

Let us now consider this problem from a different point of view. If we use a *quasistatic model* in the calculations it may be favorable to also *think* in terms of this model and not in terms of the full set of Maxwell's equations. This is analogous to what we often do in other fields of physics. For example we use classical mechanics without worrying about quantum mechanics, Newtonian kinematics without referring to special relativity or Newton's law of gravitation without thinking about general relativity. It is in the solenoid example easy to calculate the magnetic and electric fields using quasistatics. One gets an exact solution within the quasistatic model and a good approximation to the true Maxwell solution. This quasistatic solution (of MQS and Darwin) has no magnetic field at all outside the solenoid. This solution is fundamentally inconsistent with Maxwell's equations since there is a time-varying E-field in free space where the magnetic field vanishes exactly. However, the solution is of course consistent with quasistatics. This is an indication of the quite different nature of quasistatic models and Maxwell's equations. The PDE's appearing in quasistatics are often elliptic or parabolic while the Maxwell equations are hyperbolic. The conceptual problems have their roots in using *quasistatic calculations* but interpreting the result using *Maxwell's equations*.

VI. ELECTROMAGNETIC SIMULATIONS

A. The PC in basic courses

Electromagnetic simulations are essential to electrical and electronic product designs in many industries. A broad range of important applications are within the quasistatic regime like motors, sensors, power generators, transformer systems and Micro Electro Mechanical Systems (MEMS). Industrial and scientific applications may involve extensive calculations and accordingly the

need for much computational power. However, the usual PC has developed enormously and so has the software for simulations with increasingly user friendly interfaces. Today the PC is a sufficiently powerful tool for modeling many electromagnetic phenomena within just seconds of computational time. This makes it potentially useful as an instrument for teaching basic electrodynamics. In the PC lab students may simulate various electromagnetic phenomena. Examples within statics or quasistatics include the edge effects in the parallel plate capacitor, distributed currents in conductors of various shapes, the Hall effect, magneto resistance, the appearance of non vanishing charge density in inhomogeneous conductors, the electric field outside and surface charge on a conductor with currents, various objects placed in a given external static or time-varying electromagnetic fields, eddy currents, inductive heating, magnetic diffusion, magnetic shielding and more.

The use of electromagnetic simulations as an effective pedagogical tool also presupposes some support from theory. One should in particular focus more on how Maxwell's equations are used to formulate well-posed PDE-problems for various physical situations. Most textbooks, at least the basic ones, are not much influenced by the appearance of computers. Their guiding principle is to find analytical solutions in simple and instructive cases. The underlying PDE-problem may not be needed explicitly in some examples like when integral laws are used in very symmetric cases or when tricks like mirror charges or mirror currents work. When the PDEs are actually used, and this is much more the case in the more advanced texts, one finds solutions in terms of integrals and series by comparatively tedious calculations involving, for example, special functions and variable separation.

With a numerical PDE-solver available the situation is somewhat different. The formulation of well posed PDE-problems for various phenomena is now motivated without the intention of finding analytical solutions. This is an easy part of the PDE theory for electromagnetics and it is instructive by providing increased insight into the structure of Maxwell's equations. It is in particular important and sometimes straightforward to find how Maxwell's equations reduce thanks to various symmetries. For example, the independence of one Cartesian coordinate or alternatively an axial symmetry results in a decoupling of Maxwell's equations. A simple inspection of the equations written in component form reveals this structure. Such symmetries explain, for example, why the magnetic field of a toroidal coil only has an azimuthal component or why there are TE and TM modes in planar wave-guides. In the next subsection we illustrate this by formulating PDE problems for eddy currents.

B. Equations for eddy currents

Eddy currents and associated phenomena like inductive heating, magnetic shielding and magnetic diffusion are in practice very common since the basic ingredients are just a conductor and a time-varying electromagnetic field. Still eddy currents are not much discussed in the textbooks. An exception is the one by Smythe where a whole chapter is devoted to eddy currents and analytical solutions are given in terms of series and integrals.³⁷ An interesting method of images may be used to calculate eddy currents in a thin conducting sheet. The theory was developed by Maxwell and reformulated in modern terms by Saslow.³⁸ However, the analytical theory for eddy currents is still comparatively difficult. Electromagnetic modeling on a PC is today an attractive and easy way to include more about eddy currents in basic courses.

Eddy current problems fall into two classes, steady-state and transient. In steady state analysis (also called time-harmonic analysis) we simply replace $\partial/\partial t$ with the factor $j\omega$ and allow for complex valued fields. Maxwell equations becomes time-independent and much easier to solve. Physically we may obtain a steady-state condition sufficiently long time after the start of a time-harmonic source of the fields. The initial transient behavior is not considered in this analysis. The steady state Maxwell equations may be further simplified in quasistatic situations, this may be useful for analytical theory but is not of much interest for electromagnetic simulations. However, in time-dependent (transient) analysis an initial value PDE must be solved. Then a quasistatic approximation may imply a major numerical simplification changing a hyperbolic PDE into an parabolic or elliptic one.

Let us assume that the time-variation is slow enough so that quasistatic theory applies. But what quasistatic model should we use in the study of eddy currents; EQS, MQS or Darwin? Obviously not EQS since magnetic induction (Faraday's law) is outside the scope of that model. What about MQS? This seems to be the standard model for eddy currents calculation but it may in fact only be used in some particular cases.^{15,37,39} A quite common misunderstanding concerning the quasistatic approximation may partly explain the popularity of MQS.²⁷

Let us now consider situations where the use of MQS may be justified. This happens in certain symmetric cases when MQS and Darwin are equivalent models. We consider below two examples of such symmetries. The first one include, as a special case, the situation when Maxwell's theory for eddy currents in thin conducting sheets applies.³⁸

Example 6 Consider a body in which the conductivity $\sigma = \sigma(z)$ only varies in the z -direction. In the xy -directions the body is homogeneous and of "infinite" extent. The electromagnetic field is created by the use of an external current density $\mathbf{J}^e = J_x^e(t, x, y, z)\hat{\mathbf{x}} + J_y^e(t, x, y, z)\hat{\mathbf{y}}$ with vanishing divergence; $\nabla \cdot \mathbf{J}^e = 0$.

In this example there will appear no charge density

and the Darwin model is equivalent to MQS. The reason is that electric fields will only appear in directions of constant conductivity (i.e. they have no z -component) and they cause no pile up of charge. The system may be described in terms of the potentials A_x and A_y while $V = A_z = 0$. From equation (25) with $\mathbf{J} = \mathbf{J}^e + \sigma(-\nabla V - \partial\mathbf{A}/\partial t)$ we get the following equations for A_x and A_y

$$\begin{aligned} -\mu_0\sigma\frac{\partial A_x}{\partial t} + \nabla^2 A_x &= -\mu_0 J_x^e \\ -\mu_0\sigma\frac{\partial A_y}{\partial t} + \nabla^2 A_y &= -\mu_0 J_y^e \end{aligned} \quad (55)$$

These 3D equations determine the dynamics and may be used to model physical systems with the above symmetry.

A similar 2D axi-symmetric case may also be formulated.

Example 7 We use cylindrical coordinates (r, ϕ, z) and a rotationally symmetric conducting body with conductivity $\sigma = \sigma(r, z)$. The external current density is $\mathbf{J}^e = J^e(t, r, z)\hat{\phi}$.

Also in this case there will appear no charge density. Some potentials vanishes, $V = A_r = A_z = 0$, and the system may be described in terms of the vector potential $\mathbf{A} = A(t, r, z)\hat{\phi}$. This time we get the equation for the dynamics as

$$-\mu_0\sigma\frac{\partial A}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) + \frac{\partial^2 A}{\partial z^2} = -\mu_0 J^e \quad (56)$$

The equations may easily be studied by using a PC and some PDE-solver.⁴⁰ For example, a straight copper wire above a copper plate is considered by Backstrom using equation (55) and the solver FlexPDE.⁴¹ He also considers a similar example using (56) for a circular copper wire above a circular copper plate. The transient fields appearing when a current is turned on in the wire is studied (pages 75 and 99 in Backstrom's book). The same models are also easily studied using Comsols Multiphysics with the Electromagnetic Module. These problems are numerically simple and, of course, there are many other PDE solvers that may be used. Such numerical tools make it possible to consider many cases for which there are no analytical solutions.

The examples above were carefully designed in order to avoid charge density. This is also true for all eddy current examples in Saslows' paper (see in particular Appendix B of that paper).³⁸ Usually there will however appear time-varying surface charge on conductors with eddy currents and, in case of an inhomogeneous conductor, there will also appear charge density inside the conductor. In order to describe the physics we then need to include the scalar potential in the analysis and MQS cannot be used. An interesting possibility is then to use the Darwin model. Such applications of Darwin's model have been suggested in connection with eddy currents in the human body caused by high voltage transmission lines.⁴²

The traditional use of Darwin's model is not eddy currents but concerns charged particle beams and plasma simulations.

Let us write equations for the Darwin model in term of potentials. The conductivity of a possibly inhomogeneous conductor is $\sigma = \sigma(\mathbf{r})$ and that the time-dependent external current $\mathbf{J}^e = \mathbf{J}^e(t, \mathbf{r})$ is prescribed. The total current may be written $\mathbf{J} = \mathbf{J}^e + \sigma(-\nabla V - \partial\mathbf{A}/\partial t)$. We use, as always in the Darwin model, the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. The dynamics of the potentials are determined by the continuity equation

$$-\varepsilon_0\frac{\partial}{\partial t}\nabla^2 V - \nabla \cdot (\sigma\nabla V + \sigma\partial\mathbf{A}/\partial t) = -\nabla \cdot \mathbf{J}^e \quad (57)$$

and the Ampère-Darwin equation

$$-\varepsilon_0\mu_0\frac{\partial}{\partial t}\nabla V - \mu_0(\sigma\nabla V + \sigma\partial\mathbf{A}/\partial t) + \nabla^2 \mathbf{A} = -\mu_0\mathbf{J}^e \quad (58)$$

These equations have a non-standard appearance by containing mixed time and spatial derivatives. They may however be solved using Comsol Multiphysics. Actually there is a considerable freedom for the user to enter non-standard equations in this PDE-solver. This is achieved by making the equations available also on the "weak form" level which is the natural form for the finite element method. The numerical solutions appears, at least qualitatively, to behave in an expected way for the few examples that we have considered. The practical usefulness of the Darwin model for eddy current calculations remains however to be proven.

A qualitatively new feature of Darwin's model (as opposed to MQS or EQS) is the possibility of resonance. Since the Darwin model includes both capacitive and inductive phenomena it may in principle be used to model systems where the energy oscillates between the electric and magnetic fields. Consider, for example, a field theoretic version of a LC-circuit. A suitable design for such an application may be a resonator in the form of a short coaxial cable where one end is short circuited by a metal plate and the other end is still electrically open but with increased capacitance created by the use of two close parallel plates, connected to the inner and outer conductor, respectively. This kind of resonator has applications in connection with electron beam devices at microwave frequencies (example 3.4.1 in the textbook of Haus and Melcher).¹¹ The resonance frequency of a LC-circuit is $f = 1/\sqrt{LC}$ but the frequency must be low enough not to violate the quasistatic assumption. In the above design of a resonator the most essential method to achieve this (in the absence of dielectrics or magnetic materials) is to make the capacitance large and thus take the two capacitor plates very close to each other.

VII. SUMMARY

Maxwell's equations are fundamental for the description of electromagnetic phenomena and valid for an enor-

mous range of spatial and temporal scales. The static limit of the theory is well defined and of course much easier. The electric and magnetic fields may in this limit be given by the laws of Coulomb and Biot-Savart. However, as soon as there is *any* time-dependence we should *in principle* use the full set of Maxwell's equations with all their complexities. Time-retardation is a fundamentally important but also a complicating feature. Using Maxwell's equations means in analytical theory that even if the effect is small it will not vanish and this makes the theory unnecessary complicated. In numerical analysis these effects, however small, may force us to use smaller timesteps (for numerical stability) and expensive calculations. It is therefore useful to introduce quasistatic approximations in Maxwell's equations. The quasistatic models are also useful for a better understanding both of low frequency electrodynamics and for explaining the transition from statics to the general high frequency electrodynamics. This have been discussed in the present paper and below we list a few major points.

- (1) The quasistatic limit of Maxwell's equations is a kind of $c \rightarrow \infty$ limit obtained by neglecting time-retardation. The Darwin model is obtained if we use the Coulomb gauge.
- (2) The Darwin model involves both capacitive and inductive features but there is no radiation and the interactions are instantaneous. Poynting's theorem for this model shows that there is both electric and magnetic energy, but the electric energy only includes the Coulomb part of electric field.
- (3) EQS and MQS may be considered as approximations of the Darwin model. EQS includes capacitance but not inductance while MQS includes inductance but not capacitance. Poynting's theorems for these models show that there are only electric energy in EQS and only magnetic energy in MQS.
- (4) The law of Biot-Savart is valid within EQS, MQS and the Darwin model.
- (5) The law of Coulomb is of general significance for quasistatics (EQS, MQS and Darwin) as is obvious when we use the formulation in terms of force laws (subsection IIA) or use potentials (subsection IID).
- (6) The law of Ampère is *not* of general significance within quasistatics. It is valid only in MQS but not within EQS and Darwin.²⁷
- (7) Galilei invariance of quasistatics is a somewhat delicate issue. Galilei invariance structures may be defined for EQS and MQS but if we like the force to have the corresponding invariance then only the electric force is included in EQS and only the magnetic force in MQS.³⁵
- (8) Quasistatics has important applications in electromagnetic modeling of transient phenomena.

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 - ¹³ Dennis W. Hewett and John K. Boyd, "Streamlined Darwin simulation of nonneutral plasmas" J. Comput. Phys. **29**, 166-181 (1987)
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 - ¹⁵ John David Jackson, *Classical Electrodynamics*. (John Wiley & Sons, 1999) 3rd ed.
 - ¹⁶ It has always been tempting to regard $\partial \mathbf{B} / \partial t$ in Faraday's law as a source of an electric field. This has sometimes caused confusion and objections; currents and charges should be considered as *the only* sources of electromagnetic fields. The integrals expressing the fields in terms of these sources involves in general the retarded time so that the news from the sources propagates with the finite velocity c However, in quasistatic approximations the interactions are instantaneous so this objection is less valid.
 - ¹⁷ The notations in Ref.1 has been the preferred choice in this paper but here is one exception. Griffiths use a script lower case "r" instead of "R" but unfortunately I have not found a nice representation of this symbol on my computer.

- ¹⁸ The static continuity equation together with (1) and (2) constitute a quasistatic model without capacitive or inductive effects.
- ¹⁹ It is important to note that the integral (7) converges. This is evident from the law of Biot-Savart which implies that the B-field goes to zero as $1/r^2$ when $r \rightarrow \infty$.
- ²⁰ We consider physical systems where the charges and currents do not extend to spatial infinity and may use as boundary conditions that the fields approach zero sufficiently fast far away.
- ²¹ This is never a problem according to Ref. 11 but I have not been able to convince myself that this is true under all circumstances.
- ²² Ampère's law $\nabla \times \mathbf{H} = \mathbf{J}_f$ differs from (16) if there is a non-vanishing polarisation current $\partial \mathbf{P} / \partial t$. However, Ampère's law is strictly speaking a static equation and we prefer to use (16) as a time-dependent generalization with the same name.
- ²³ Like in Ref. 2 we use the names Ampere's law for equation (16) and Ampere-Maxwell for (14). Then it seems natural to use the name Ampere-Darwin for equation (17).
- ²⁴ P.A. Mello, "A remark on Maxwell's displacement current," Am. J. Phys. **40**, 1010-1013 (1972)
- ²⁵ T. Biswas, "Displacement current-A direct derivation" Am. J. Phys. **56**, 373-374 (1988)
- ²⁶ N. Gauthier, "A direct derivation of the displacement current", Am. J. Phys. **56**, 871-872 (1988)
- ²⁷ There seems to be a quite widespread misunderstanding that quasistatics, defined by the omission of time-retardation, includes Ampère's law (16) as a valid equation. This is stated or implied by many textbooks. See for example Jackson (Ref. 15, p. 218) or Smythe (Ref. 37, p. 368). However, this is not correct for an obvious reason. Ampère's law implies stationary current and thus constant charge density. But certainly, a slowly varying charge density cannot force us to include time-retardation.
- ²⁸ Ernst Breitenberger, "Magnetic Interactions between Charged Particles." Am. J. Phys. **36**, 505-515 (1968)
- ²⁹ Oleg D. Jefimenko, *Electricity and Magnetism* (Appleton-Century-Crofts, New-York, 1966)
- ³⁰ Allan N. Kaufman and Peter S. Rostler, "The Darwin Model as a Tool for Electromagnetic Plasma Simulation." Phys. Fluids **14**, 446-448 (1971)
- ³¹ J. D. Jackson, "From Lorenz to Coulomb and other explicit gauge transformations." Am. J. Phys. **70**, 917-928 (2002)
- ³² O. Heaviside, "On the electromagnetic effect due to the motion of electrification through a dielectric" Philos. Mag. Ser. 5 **27**, 324-339 (1889). The Darwin Lagrangian in Ref. 19 was, as mentioned in Ref. 20, obtained already here
- ³³ C. G. Darwin, "The dynamical motion of charged particles." Philos. Mag. Ser. 6 **39**, 537-551 (1920)
- ³⁴ Max Jammer and John Stachel, "If Maxwell had worked in between Ampère and Faraday: An historical fable with a pedagogical moral" Am. J. Phys. **48**, 5-7 (1980)
- ³⁵ M. Le Bellac and J.-M. Lévy-Leblond "Galilean electromagnetism" NuovoCimento **14B**, 217-233 (1973)
- ³⁶ The term $\partial \mathbf{E}_C / \partial t$ goes to zero as $1/r^3$ when $r \rightarrow \infty$.
- ³⁷ William R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill, New York, 1968), 3rd ed.
- ³⁸ W. M. Saslow, "Maxwell's theory of eddy currents in thin conducting sheets, applications to electromagnetic shielding and MAGLEV." Am. J. Phys. **60**, 693-711 (1992)
- ³⁹ Eckhard Baum and Otto Erb, "Eddy currents and electrical surface charge" Int. J. Numer. Model. **16**, 199-218 (2003).
- ⁴⁰ There are many PDE-solvers that are suitable for electromagnetic simulations in basic courses. In the textbook by Gunnar Backstrom (Ref. 41) the program FlexPDE is used. I am using COMSOL Multiphysics with the Electromagnetic module. Information about these programs may be found on the homepages <http://www.pdesolutions.com/> and <http://www.comsol.com/>.
- ⁴¹ Gunnar Backstrom, *Waves by Finite Element Analysis* (Studentlitteratur, Lund, 1999).
- ⁴² Pierre-Arnaud. Raviart and Eric Sonnendrücker, "Approximate models for the Maxwell equations" J. Comput. Appl. Math. **63**, 69-81 (1994)